Lecture 8

Plan:

- Faces of Polyhedra
- State tons of facts
- Prove them

Faces of Polyhedra

Def: $a^{(1)}, \dots, a^{(k)} \in \mathbb{R}^n$ are <u>affinely independent</u> if

$$\sum_{i=1}^{k} \lambda_i a^{(i)} = 0$$

and $\sum \lambda_i = 0$ imply $\lambda_1 = \dots = \lambda_k = 0$.

(w/out $\sum \lambda_i = 0$, is just linear indp.)

linear independence $\Rightarrow$ affine independ.

<u>Note</u>: $\mathrm{aff}(X)$ = lowest dim. affine space containing X.

$\{a^{(i)}\}$ affinely independent iff

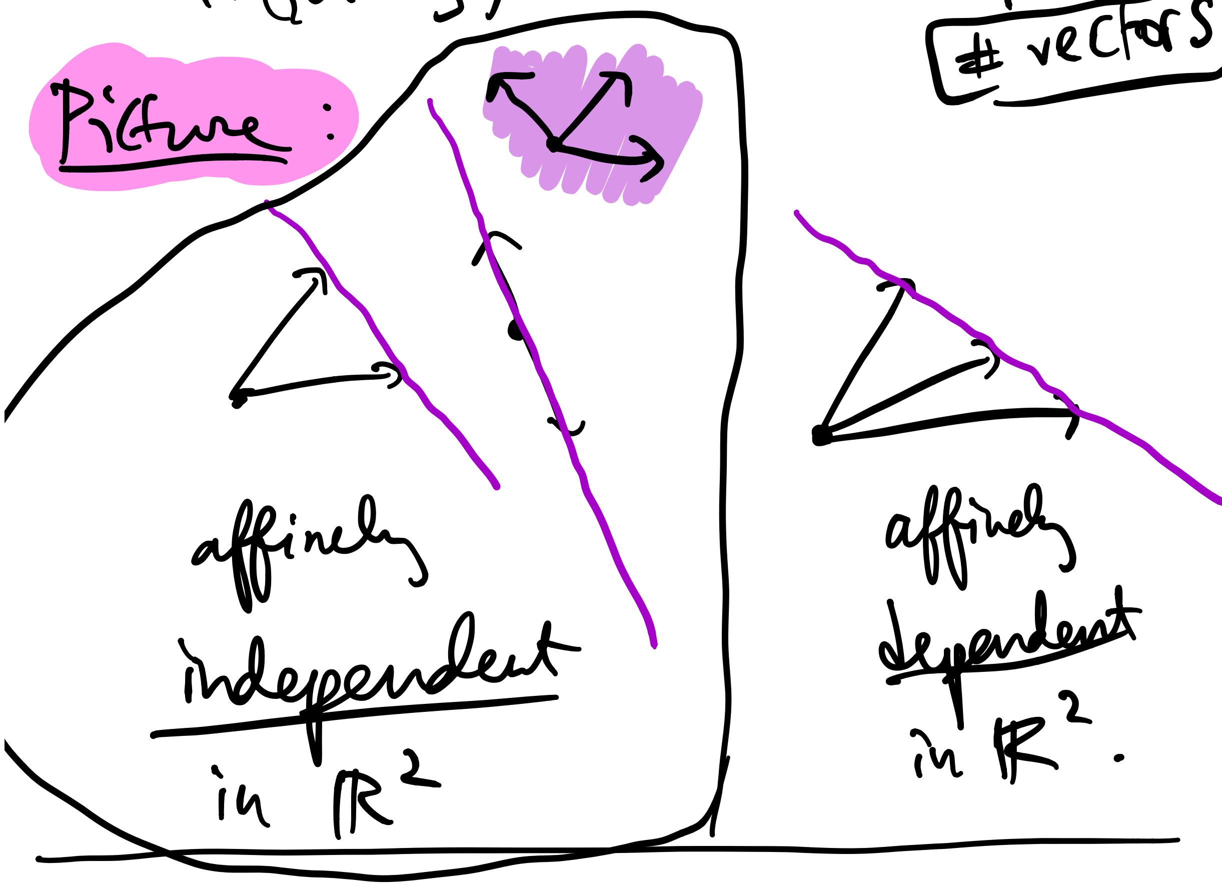
$\{a^{(i)}\}$ affinely independent

$\left\{ \begin{bmatrix} a^{(i)} \\ 1 \end{bmatrix} \right\}$ linearly independent.

$\Leftrightarrow$ $\mathrm{aff}(\{a^{(i)}\})$ has dimension $k-1$

↑ # vectors.

Picture:



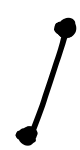
Def Dimension $\dim(P)$ of polyhedron P:

$-1 +$ max # affinely independent points in P.

Equivalently, dimension of affine hull $\mathrm{aff}(P)$.

Examples: $P = \emptyset$, $\dim(P) = -1$

P = singleton  $\dim(P) = 0$

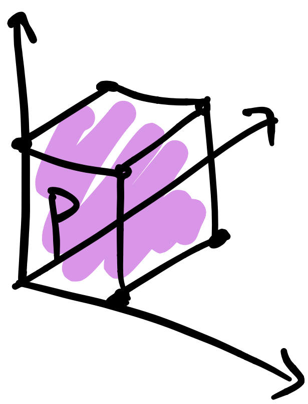
P = line segment  $\dim(P) = 1$

$\vdots$

$\mathrm{aff}(P) = \mathbb{R}^n$ $\dim(P) = n$;

P "full dimensional"

e.g. cube in $\mathbb{R}^3$: $\{x : 0 \le x_i \le 1\}$



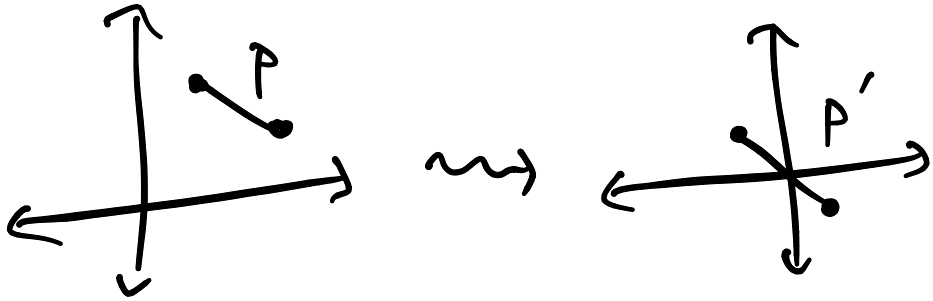
$\dim P = 3$

$\dim \mathbb{R}^3 = 3$ (as polyhedron).

Why affine, not linear? affine independence is translation invariant :

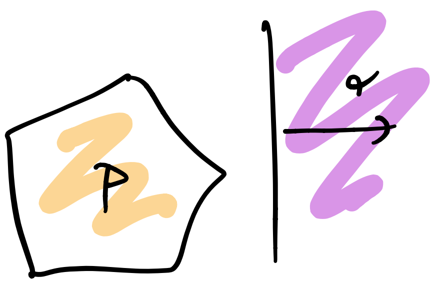
if I used max # lin indep points − 1

$\dim(P) = 1$ $\dim(P') = 0$

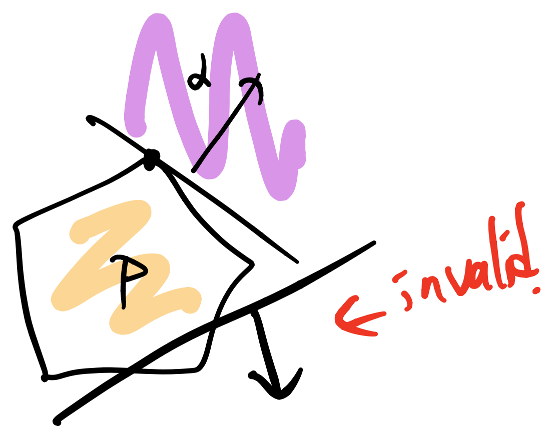


$1 = \dim(P) = \dim(P')$.

Def: $\alpha^T x \leq \beta$ is a <u>valid inequality</u> for P if $\alpha^T x \leq \beta$ for all $x \in P$.



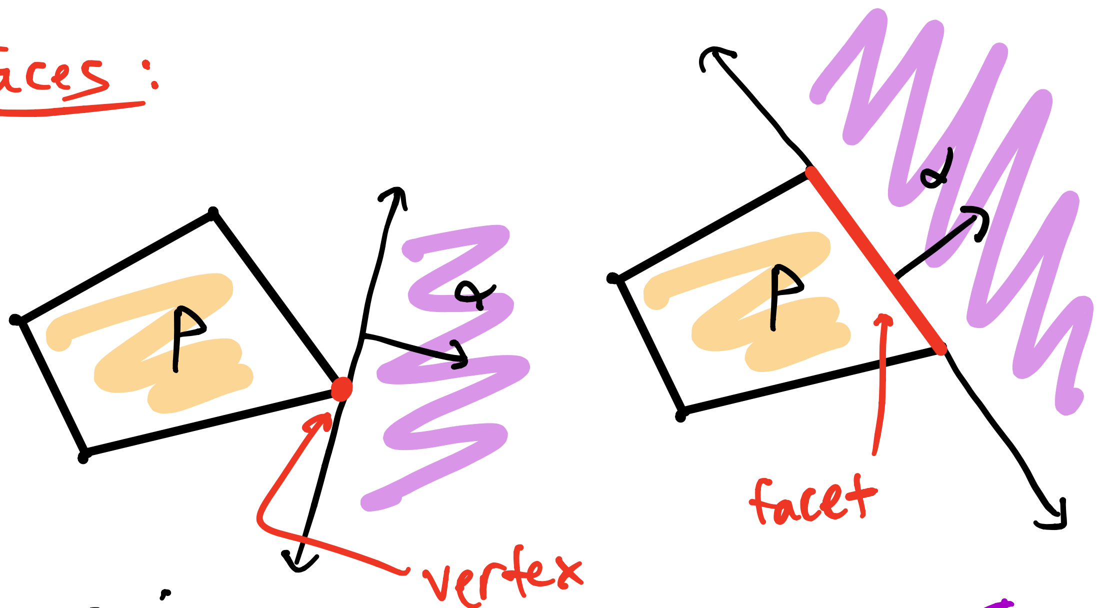
or



Def A <u>face</u> of a polyhedron P is $\{x \in P : \alpha^T x = \beta\}$ for

$\alpha^T x \leq \beta$ valid.

Faces:



Properties:

$0^T x = 0$

- Faces are polyhedra
- Empty face & entire P are called trivial faces

dim = -1   dim P

- else F nontrivial

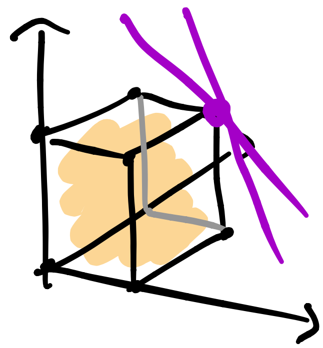
$$0 \leq \dim(F) \leq \dim P - 1$$

- $F : \dim(F) = \dim(P) - 1$ called facets.
- $F : \dim(F) = 0$ called vertices

• F : $\dim(F) = 0$ called vertices

Ex: list the 28 faces of the cube

$$P = \{x \in \mathbb{R}^3 : 0 \le x_i \le 1\}$$

$P =$ 

Fact: ∞ many valid ineqs, but # faces finite!

EVERYTHING ABOUT POLYHEDRA

$$A = \begin{bmatrix} — & a_1^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} \in \mathbb{R}^{m\times n} \qquad P = \{x : Ax \le b\} \subseteq \mathbb{R}^n$$

- **Face Characterization:**

Any nonempty face of P is

$$F_I = \left\{ x : \begin{array}{ll} a_i^T x = b_i & \forall i \in I \\ a_i^T x \le b_i & \forall i \notin I \end{array} \right\}$$

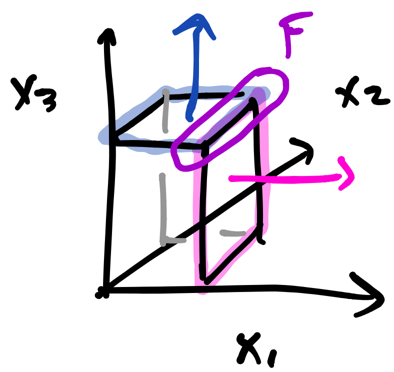
for some set $I \subseteq \{1,\dots,m\}$.

(and F_I is always a face)

$\Rightarrow$ intersection of faces is face

⇒ An intersection of faces is face

E.g. cube



$$F = \left\{ x : \begin{array}{c} x_3 = 1 \\ x_1 = 1 \\ 0 \le x_2 \le 1 \end{array} \right\}$$

⇒ # faces $\le 2^m + 1$

- **Facet Maximality:** The facets are the maximal nontrivial faces of a nonempty polyhedron P.

For vertices: just need equalities.

vertices = extreme points. Exercise

③ Vertex Characterization:

Suppose x^* extreme point of P

Then $\exists I$ s.t. x^* is the unique soln to

$$* \quad \boxed{a_i^T x = b_i \text{ for all } i \in I}$$

Moreover any $x \in P$ that uniquely solves $*$ is extreme.

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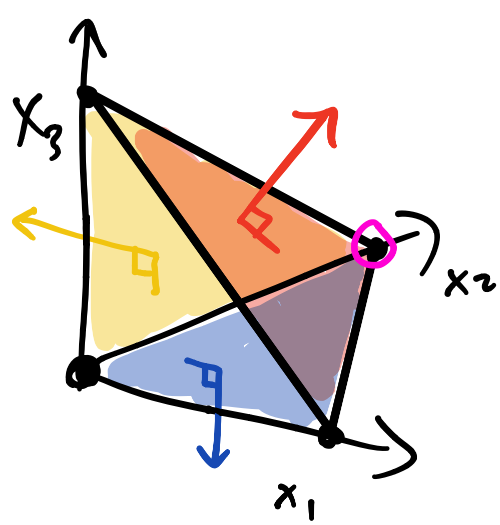
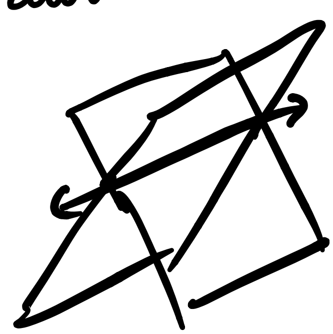
e.g. simplex $(0,1,0)$ is intersection of 3 constraints





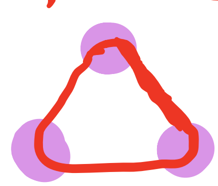
- **Vertex minimality:** For rank$(A)=n$, minimal nontrivial faces of polyhedron P are the vertices.

Exercise: if rank$(A)<n$, no vertices!

- **Polytopes = convex hulls**

If a polyhedron P is bounded then $P=\mathrm{conv}(\{\text{extreme points of } P\})$.

(special case of Krein-Milman theorem: compact convex subset of $\mathbb{R}^n$ is conv(extreme pts).)

- **Facets Characterize** 

□ inequality $a_i^T \le b_i$ **redundant** if P unchanged when it's removed.

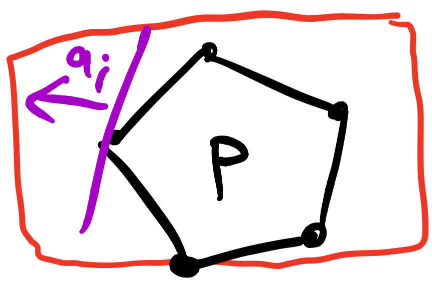
□ $I_= := \{i : a_i^T x = b_i \ \forall x \in P\}$ "equalities"

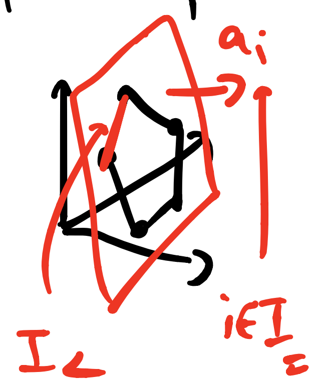
□ $I_< := \{i : \exists x \in P \ a_i^T x < b_i\}$.

"real inequalities"

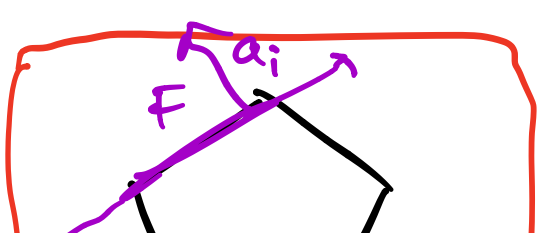
THEN:

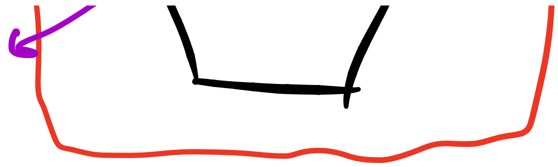
(Sufficiency:) If face $a_i^T x \le b_i$ for $i \in I_<$ is not facet, then $a_i^T x \le b_i$ is redundant.



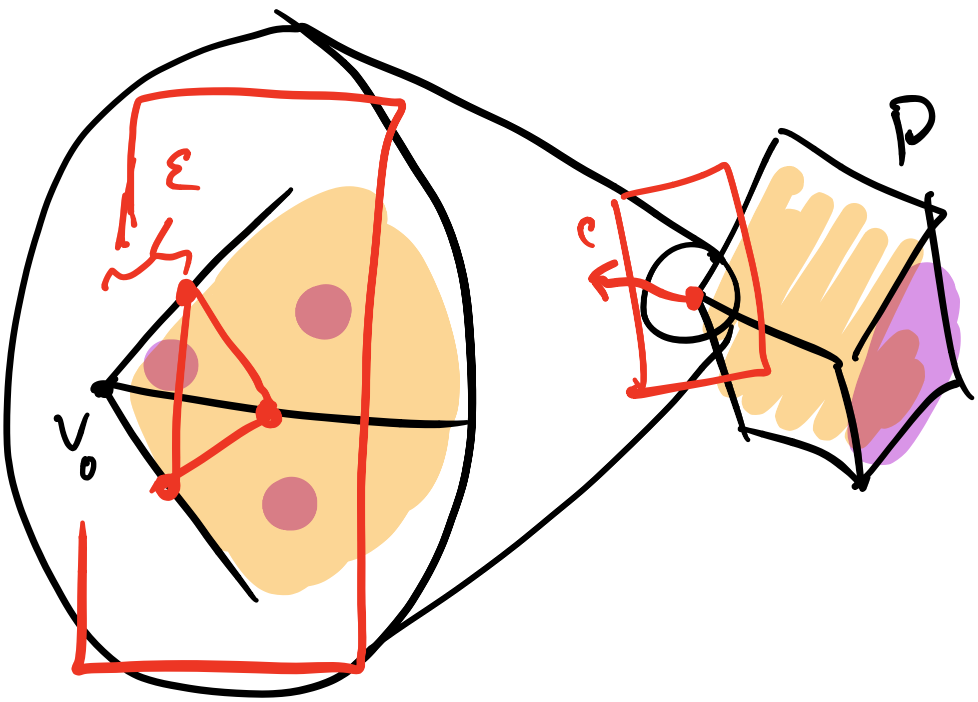


(Necessity:) If F is facet of P, $\exists\, i \in I_<$ such that F is induced by $a_i^T x = b_i$





- <u>Near vertices =</u>
  <u>cones over polytopes</u>



Let v_0 vertex of P from valid inequality $c^T x \leq m$.

Let ε be such that $\boxed{c^T v' \leq m - \varepsilon}$

for all other vertices v'.

Then

$$P_0 = \{ x \in P : c^T x = M - \varepsilon \}$$

is a <u>polytope</u> & is bijection

$\{P_0$'s dim k faces$\}$

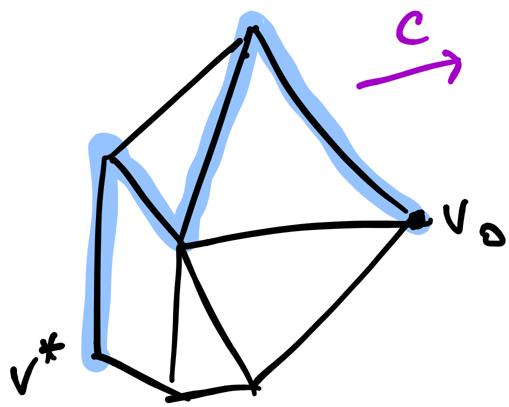
$\longleftrightarrow$

$\{P$'s dim $k+1$ faces contain $v_0\}$.

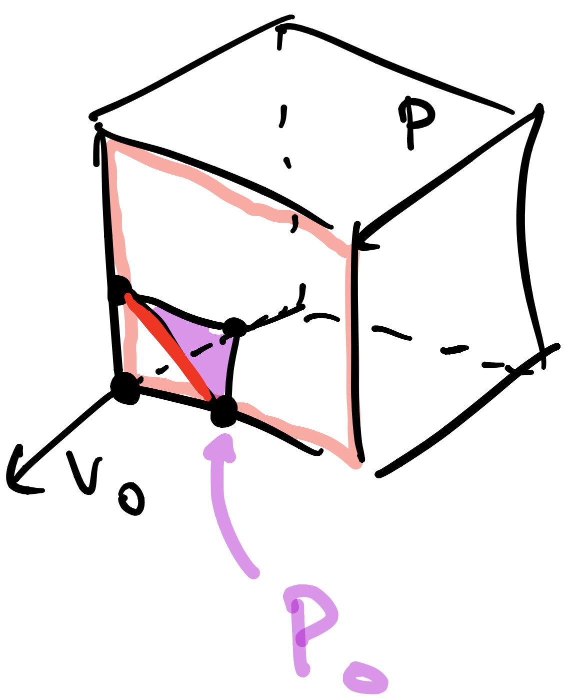
- <u>**P's "graph" connected:**</u> Graph of vertices & edges of polyhedron P

is always connected.

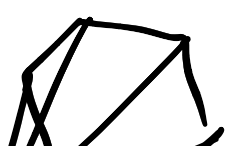
In particular: if v^* minimum of $c^T x$ over P, v_0 vertex, $\exists\ v_0 \to v^*$ path which decreases objective.



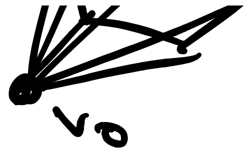
$$v_0 \vec{1} \geq 0$$



bijection: dim k face F of P_0 $\longrightarrow$ dim $k+1$ face of P_0 P containing F and v_0.



PROOFS



Recall face characterization:

Let $A \in \mathbb{R}^{m \times n}$, $\quad A = \begin{bmatrix} & \vdots & \\ - & a_i^T & - \\ & \vdots & \end{bmatrix}$

Any nonempty face of $P = \{x : Ax \leq b\}$ is

$$\left\{ x : \begin{array}{ll} a_i^T x = b_i & \forall i \in I \\ a_i^T x \leq b_i & \forall i \notin I \end{array} \right\}$$

for some set $I \subseteq \{1, \ldots, m\}$.

Proof of converse: exercise

Proof Consider valid inequality $\boxed{\alpha^T x \leq b}$ giving nonempty face F.

$F = \{x : \alpha^T x = b\} \cap P$

- F = optimum solutions to bounded LP

$$\boxed{(P) \quad \begin{array}{l} \max \ \alpha^T x \\ \text{subject to } Ax \leq b \end{array}}$$

- Let y^* optimal solution to dual.

$y^* = (y_1, \ldots, y_m)$

- Complementary slackness:

optimal solns F are

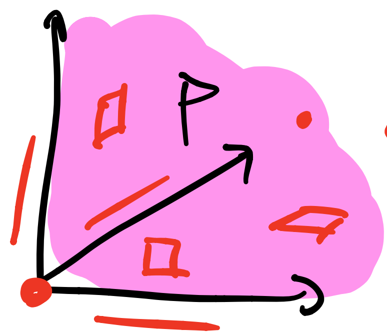
$$\boxed{\{x : a_i^T x = b_i \text{ for } i : y_i^* > 0\}.}$$

Thus we can take $I = \{i : y_i^* > 0\}$. $\square$

Ex:
- positive orthant $\{x \in \mathbb{R}^n : x_i \geq 0\} = P$ has $\boxed{2^n + 1}$ faces

  ($\uparrow$ n inequalities)

- How many of dim k? $\{x_i = 0 \;\; i \in I\}$



$\leftarrow 8 + \emptyset = 9$

For polytopes can also bound # faces in terms of # vertices.
("upper bound theorem"
"Dehn Somerville equations")

Facet maximality

Pf : Exercise to prove from face characterization.

Recall vertex characterization:

Let x^* extreme point for P.

Then $\exists \, I$ s.t. x^* is the unique soln to

$$\boxed{a_i^T x = b_i \;\; \forall i \in I.}$$

moreover, any such unique solution $x^* \in P$ is extreme.

Proof: Given extreme point x^*,

- define $I = \{ i : a_i^T x^* = b_i \}$.

- Note for $i \notin I$, $\boxed{a_i^T x^* < b_i}$.

- By "faces characterization", x^* uniquely defined by

$$F = \left\{ \begin{array}{lll} (*) & a_i^T x = b_i & i \in I \\ (**) & a_i^T x \leq b_i & i \notin I. \end{array} \right\} = \{x^*\}$$

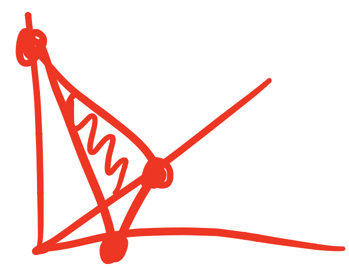
- Suppose $\exists$ other soln. $\hat{x}$ to $(*)$. (for contradiction)

- Because $\underline{a_i^T x < b_i}$ for $i \notin I$,

$$\boxed{(1-\varepsilon)x^* + \varepsilon\hat{x}}$$

still satisfies (*), (* *) for $\underline{\varepsilon \text{ small enough}}$.

- Contradicts F having only one point. $\square$.



Basic Feasible Solutions:

For $Q = \{Ax=b\ ,\ x \geq 0\}$

can describe extreme points very explicitly.

(every P can be put in this form).

Corollary of Vertex Thm: Extreme pts. of Q as above come from setting

$$\boxed{x_j = 0 \text{ for } j \in J}$$

and finding unique solution to $Ax = b$ for remaining variables.

Can say more: Extreme points of Q as above are the basic feasible solutions (bfs), feasible solns obtained as follows:

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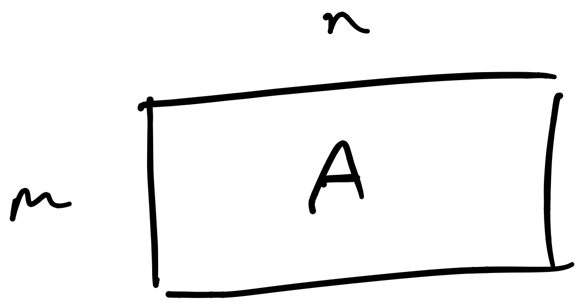
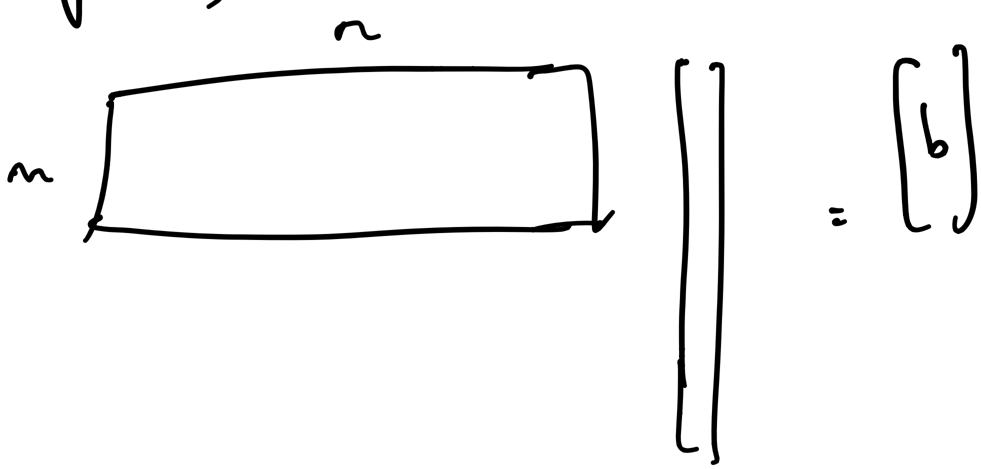
- Remove redundant rows from A (   )



- Choose m columns B of A,



- Solve $A_B x_B = 0$, set

$$x^*_i = \begin{cases} & i \in B \\ & \text{else} \end{cases}$$

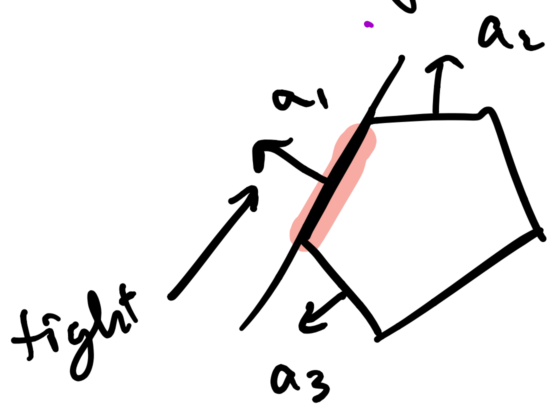
$$\boxed{\{\text{bfs}\} = \{\text{extreme pts}\}}.$$

Recall vertex minimality

If $\boxed{\text{rank } A = n}$, vertices are minimal nontrivial faces of P.

$P = \{x : Ax \leq b\}$

$$\begin{bmatrix} \vdots \\ - \; a_i^T \; - \\ \vdots \end{bmatrix}$$



Proof: Let F min'l face of P.

- Face characterization $\Rightarrow \exists I$

$$F = F_I = \left\{ x : \begin{array}{l} a_i^T x = b_i \ \forall i \in I \\ a_j^T x \le b_j \ \forall j \notin I \end{array} \right\}$$

assume no redundant inequalities incl.

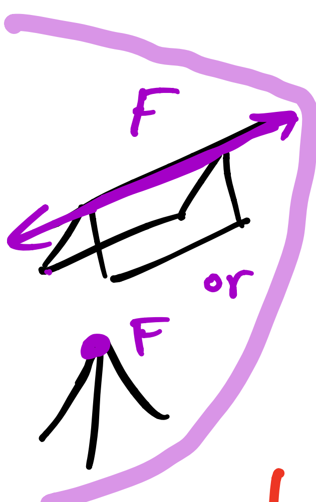
and adding any elt to I makes F_I empty. (b/c else F_I face $\subsetneq F$).

- Consider two cases:

(a) Only the equalities are needed ($a_j^T x \le b_j$ redund. for F).

i.e. F is exactly

$$\{x : a_i^T x = b_i \ \forall i \in I\}.$$



☆ Claim: $\forall j \notin I$,

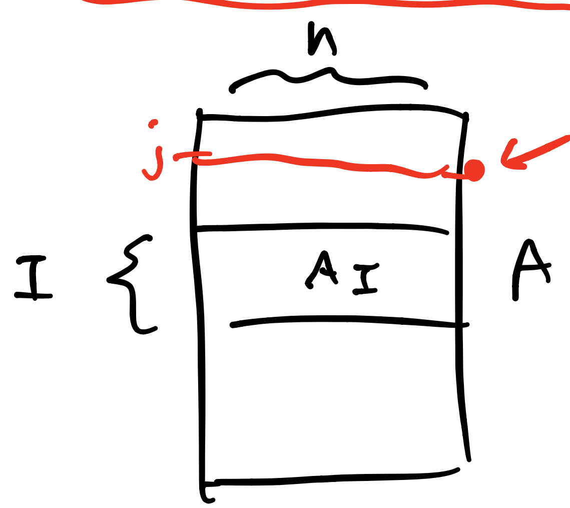
$$a_j \in \text{lin}(a_i : i \in I);$$

else $a_j^T x = b_j + 1$ has solution in F, contradicting $a_j^T x \le b_j$.

($a_i^T x : i \in I$ do not determine $a_j^T x$ unless $a_j \in \text{lin}(a_i : i \in I)$)

★ Equivalently: submatrix A_I w/ rows in I satisfies

$$\boxed{\text{row}(A) = \text{row}(A_I)} \, ;$$

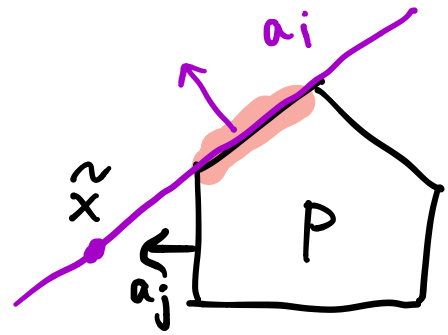


bc if $\text{row}(A) \not\ni a_j$, then can choose $a_j^T x$ freely for $x \in F$. ★

hence $\boxed{\text{rank}(A_I) = \text{rank}(A) = n}$.

★ Thus: $a_i^T x = b_i$ for $i \in I$ has unique soln, so F is single point, i.e. a vertex. ✓

(b) Some inequality needed;
we'll show is contradiction.



- $\exists\, j \notin I$, $\tilde{x}$ w/ $(\tilde{x} \notin P)$

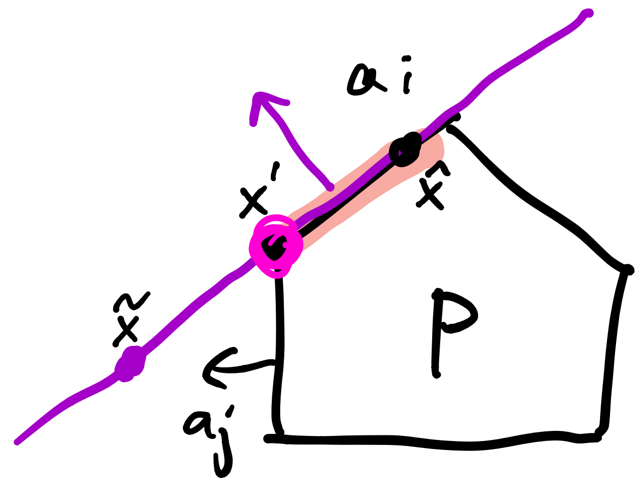
$$\boxed{\begin{array}{l} a_i^T \tilde{x} = b_i \quad i \in I, \\ a_j^T \tilde{x} > b_j \end{array}}$$

- F nontrivial $\Longrightarrow \exists\, \hat{x} \in F$.

$\hat{x}$ satisfies
$$\boxed{\begin{array}{l} a_i^T x = b_i \quad i \in I, \\ a_j^T \hat{x} \leq b_j \end{array}}$$

- Consider convex combination

$$\boxed{x' = \lambda\tilde{x} + (1-\lambda)\hat{x}}$$



let λ be largest so $x' \in P$. ($\in F$).

- x' satisfies <u>one more equality</u>

(else could increase λ)

contradicts minimality of F. $\square$

Finally we can show equiv b/w bounded polyhedra & convex hulls. (polytopes).

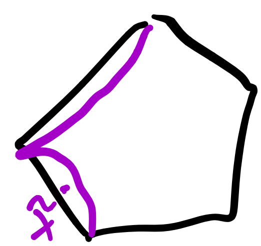
Recall: $P = \{Ax \le b\}$ bounded then $P = \text{conv}(\underbrace{\text{Xtreme pts. of } P}_{X})$.

!!

i.e. P is polytope.

Proof: Use TOTA.

- $X \subseteq P \Rightarrow \text{conv}(X) \subseteq P$.
- Assume for contradiction that $\text{conv}(X) \subsetneq P$.
- Let $\tilde{x} \in P \setminus \text{conv}(X)$.



- Then

$$\sum_{v \in X} \lambda_v v = \tilde{x}$$

$$\sum_{v \in X} \lambda_v = 1$$

$$\lambda_v \geq 0$$

has no solution.

• TOTA $\Rightarrow$

$$\tilde{A} \to \begin{bmatrix} \cdots & \vert\ v\text{'s}\ \vert & \cdots \\ 1\,1\,1 & \cdots & 1\,1\,1 \\ & -I & \end{bmatrix} \begin{bmatrix} \vert \\ \lambda \\ \vert \end{bmatrix} \underbrace{\begin{matrix} = \\ = \\ \leq \end{matrix}}_{\Delta} \overbrace{\begin{bmatrix} \tilde{x} \\ 1 \\ 0 \end{bmatrix}}^{\tilde{b}}$$

$\Delta \rightsquigarrow \square$

$\leq \to \geq$

$= \to ?$

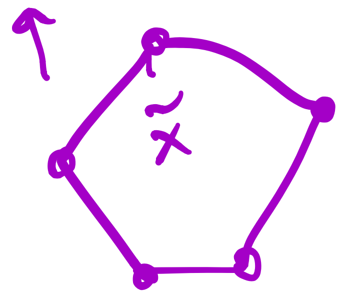
has no soln $\Leftrightarrow$

$$\boxed{\tilde{A}^T y = 0,\ \tilde{b}^T y < 0,\ y \,\square\, 0}$$

has soln. i.e.

$\tilde{A}^T$ y

$$\begin{bmatrix} \vdots & 1 & \\ -v\text{'s}- & \vdots & -I \\ \vdots & 1 & \end{bmatrix}\begin{bmatrix} c\\ t\\ s\end{bmatrix} = 0 \qquad (*)$$



$$y \geq 0 \;\Downarrow$$

for $s \geq 0$, and $\underbrace{y^T\tilde{b} < 0}_{(**)}$.

I.E.

$(*)$ $\quad t + c\cdot v \geq 0 \quad \forall v \in X$

$(**)$ $\quad t + c\cdot \tilde{x} < 0. \Rightarrow c\cdot\tilde{x} < c\cdot v \quad \forall v \in X$

- P bounded $\Rightarrow$

$$\min\{c^T x : x \in P\} = z^* > -\infty.$$

- Face induced by $\boxed{c^T x = z^*}$ nonempty, but contains no vertex.

(because ~~$\not\exists$~~ $\Rightarrow$ objective is less on $\tilde{x}$ than any vertex.)

- Contradicts vertex minimality! ✓

orig A

(~~ ~~ applies b/c rank $A = n$;

if rank $A < n$, P not bounded (b/c some solution to $Ay = 0$. ~~ ~~)

$\square$.

assume w/log $0 \in P \Rightarrow b_i \geq 0$